Lecture 7 27-09-18

Recall: Last time we have considered

$$ay'' + by' + cy = 0 \quad \text{with } a \neq 0.$$

Summary: consider characteristic equation

$$ar^2 + br + c = 0$$

Case i) $b^2 - 4ac > 0$, real distinct root r_1, r_2

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Case ii) $b^2 - 4ac < 0$, distinct complex root

$r = \lambda + i\mu$, $\bar{r} = \lambda - i\mu$

$$y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t)$$

Case iii) $b^2 - 4ac = 0$, repeated root r

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$

§ Reduction of order:

In the case iii), we can also use the following technique called reduction of order to find another solution besides e^{rt}.

- Say we consider

$$y'' + p(t)y' + q(t)y = 0$$

with a solution $y_1(t) \neq 0 \quad \forall\, t \in I$.

Idea: Let $y(t) = v(t)\, y_1(t)$ and try to plug in to see the required equation for v:

- $y' = v'y_1 + vy_1'$, $y'' = v''y_1 + 2v'y_1' + vy_1''$

$$y'' + p(t)y' + q(t)y = 0$$

$$\Rightarrow \quad v''y_1 + 2v'y_1' + \cancel{vy_1''}^{\,0} + p(t)(v'y_1 + \cancel{vy_1'}^{\,0}) + \cancel{q(t)\,vy_1}^{\,0} = 0,$$

$$\Rightarrow \quad y_1 v'' + 2v'y_1' + p(t)\, v'y_1 = 0$$

$$\Rightarrow \quad v'' + \left(\frac{2y_1'}{y_1} + p(t)\right) v' = 0.$$

Rewrite $z = v'$:

$$z' + \left(\frac{2y_1'}{y_1} + p(t)\right) z = 0$$

which is a 1^{st}-order ODE!

$$\Rightarrow \quad z = C\exp\left(-\int \frac{2y_1'}{y_1} + p(t)\, dt\right).$$

$$= C y_1^{-2} \exp\left(-\int p(t)\, dt\right)$$

chosen be 1

$$\text{Hence} \quad v' = z = \cancel{C} y_1^{-2} e^{-\int p(t)dt}$$

taken to be 0

$$\Rightarrow v = \left(\int y_1^{-2} e^{-\int p(t)dt}\, dt\right) + \cancel{C'}$$

$$\text{Hence} \quad y_2 = y_1 \left(\int y_1^{-2} e^{-\int p(t)dt}\, dt\right)$$

is another solution to the ODE

We compute:

$$W(y_1, y_1 v) = \det\begin{pmatrix} y_1 & y_1 v \\ y_1' & y_1' v + v' y_1 \end{pmatrix}$$

$$= y_1^2 v' = e^{-\int p(t) dt}$$

Rk: Complex-valued equation:

- We may talk about

$$y'' + p(t) y' + q(t) y = 0 \quad \text{with both}$$

$p(t)$, $q(t)$ being complex-valued, and solve complex-valued $y(t)$.

It's properties is the same with real-valued eqt with very similar proof, we state the results:

Thm: (existence and uniqueness)

$$\begin{cases} y'' + p(t) y' + q(t) y = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_1 \end{cases}$$

for $y_0, y_1 \in \mathbb{C}$, $\exists!$ solution y on I.

Thm: (superposition) $c_1 y_1 + c_2 y_2$ solution for $c_1, c_2 \in \mathbb{C}$

hence $\mathcal{S}_{\mathbb{C}} := \{ y \mid y'' + py' + qy = 0 \}$ is a $\mathbb{C}$-vector sp.

Def: We let $W(y_1, y_2)(t) := \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$

for two complex-valued solution y_1, y_2

Thm: We have $W(t) := W(y_1, y_2)(t)$, which will satisfy the equation

$W'(t) + p(t) W(t)$ which can be solved

as $W(t) = c \exp\left(-\int p(t)\,dt\right)$ with $c \in \mathbb{C}$.

Thm: (Wronkian & Fundamental set of solution)

TFAE:

1, y_1, y_2 $\mathbb{C}$-Linearily independent

i.e. $c_1 y_1 + c_2 y_2 = 0 \Rightarrow c_1, c_2 = 0$
for $c_1, c_2 \in \mathbb{C}$

2, $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$

3, $W(y_1, y_2)(t) \neq 0$ for all $t \in I$

If any of (1)-(3) holds, then
we have $\mathcal{S} = \{c_1 y_1 + c_2 y_2 \mid c_1, c_2 \in \mathbb{C}\}$

i.e. y_1, y_2 is a basis for $\mathcal{S}_{\mathbb{C}}$

Rk:

- In the previous lecture, when we consider
  $ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}, a \neq 0$
  with $b^2 - 4ac < 0$ (i.e. the case for distinct $\mathbb{C}$-root)

- we have $y_1(t) = e^{rt}$, $y_2(t) = e^{\bar{r}t}$
  with $r = \lambda + i\mu$

$$W(y_1, y_2)(t) = \det\begin{pmatrix} e^{rt} & e^{\bar{r}t} \\ re^{rt} & \bar{r}e^{\bar{r}t} \end{pmatrix}$$

$$= \bar{r} e^{2\lambda t} - r e^{2\lambda t} = -2i\mu e^{2\lambda t} \neq 0.$$

$\Rightarrow$ y_1, y_2 is a fundamental set of sol.

- Now suppose $y(t)$ is a real-valued solution to $ay'' + by' + cy = 0$

$$\exists\ c_1, c_2 \in \mathbb{C},\quad y(t) = \mathrm{Re}\,(\underbrace{c_1}_{=a+ib} e^{rt} + \underbrace{c_2}_{=c+id} e^{\bar{r}t})$$

$$= e^{\lambda t}\big((a+c)\cos\mu t - (b+d)\sin\mu t\big).$$

- Observe that $e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t$ is another basis for $S_{\mathbb{C}}$ as complex vector space.

- $$S_{\mathbb{R}} = \{\, y \mid y \in S_{\mathbb{C}},\ \bar{y} = y \,\}$$

  $\Rightarrow$ $e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t$ is a set of basis for $S_{\mathbb{R}}$.

Rk: • If we only interested in $\mathbb{C}$-valued solution for $ay'' + by' + cy = 0$, $a, b, c \in \mathbb{C}$ there is only 2 cases:

i) two distinct root $r_1 \neq r_2$

and $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

ii) repeated root r

and $y(t) = c_1 e^{rt} + c_2 t e^{rt}$.

for $c_1, c_2 \in \mathbb{C}$